Signal Analysis and Communication ECE355

Chr. 2.3: Properties of LTI Systems (contd.)

Lecture 11

02-09-2023

Ch. 2.3: PROPERTIES OF LTI SYSTEMS (contd.)

⑦ LTI Stability

Recall: A system is BIBO stable if $|x_{(t)}| < \infty$ for all time results in $|y_{(t)}| < \infty$ for all time.

- And the LTI system is characterized by impulse response $h_{(t)}$

$$y_{(t)} = \int_{-\infty}^{\infty} x_{(\tau)} h_{(t-\tau)} d\tau$$

- Suppose $|x_{(t)}| < B$, $\forall t$, for some constant B, then

$$|y_{(t)}| = \left| \int_{-\infty}^{\infty} x_{(\tau)} h_{(t-\tau)} d\tau \right|$$

$$= \left| \int_{-\infty}^{\infty} h_{(\tau)} x_{(t-\tau)} d\tau \right|$$ Convolution's Commutative property.

$$\leq \int_{-\infty}^{\infty} |h_{(\tau)}| |x_{(t-\tau)}| d\tau$$ Cauchy-Schwarz Inequality.

- here $|x_{(t-\tau)}| \leq B$ for all values of $(t-\tau)$

$$\Rightarrow |y_{(t)}| \leq B \underbrace{\int_{-\infty}^{\infty} |h_{(\tau)}| d\tau}_{\text{If finite then } |y_{(t)}| < \infty}$$

THEOREM: $\boxed{\text{A CT LTI sys. is BIBO stable if } \underbrace{\int_{-\infty}^{\infty} |h_{(t)}| dt < \infty}_{\text{absolutely integrable}}}$

- Similar is true for a DT LTI system.

$$\boxed{\text{A DT LTI sys. is BIBO stable if } \underbrace{\sum_{n=-\infty}^{\infty}|h_{[n]}|<\infty}_{\text{absolutely summable}}}$$

- Conversely,
  - Suppose $\int_{-\infty}^{\infty}|h_{(\tau)}|d\tau=\infty$
  - & let $x_{(t)}=\begin{cases}0 & \text{if } h_{(-t)}=0\\ \underbrace{\frac{h_{(-t)}}{|h_{(-t)}|}}_{\text{sign of } h_{(-t)}} & \text{if } h_{(-t)}\neq 0\end{cases}$ — ①
  - Basically Eqn. ① $\Rightarrow |x_{(t)}|\leq 1$, $\forall t$
  - Let's evaluate $y_{(0)}$ [for similicity - to prove]

$$y_{(0)}=\int_{-\infty}^{\infty}h_{(\tau)}\,x_{(0-\tau)}\,d\tau$$

$$=\int_{-\infty}^{\infty}h_{(\tau)}\cdot\frac{h_{(\tau)}}{|h_{(\tau)}|}\,d\tau$$

$$=\int_{-\infty}^{\infty}|h_{(\tau)}|\,d\tau=\underset{\uparrow}{\infty}$$

Although $|x_{(t)}|<\infty$

Examples:

① $h_{(t)}=\delta_{(t-t_0)}$ CT LTI sys. characterized by $h_{(t)}$

- To check its stability:

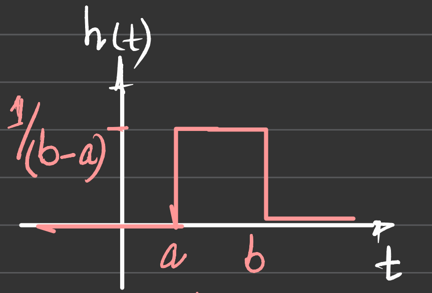
$$\underbrace{\int_{-\infty}^{\infty}|\delta_{(t-t_0)}|dt}_{\text{area of an impulse}=1}=1<\infty \quad \therefore \text{ BIBO Stable}$$

② $$h_{(t)} = \frac{1}{(b-a)}\left[u_{(t-a)} - u_{(t-b)}\right]$$

- To check it's stability:

$$\int_a^b \left|\frac{1}{(b-a)}\right| dt = 1 < \infty$$

$\therefore$ BIBO Stable

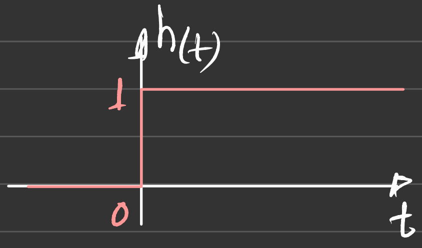


pulse exists from 'a' to 'b'.

③ $$h_{(t)} = u_{(t)}$$

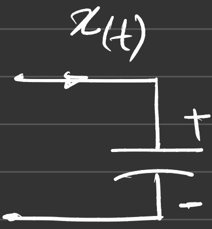
- To check stability:

$$\int_0^\infty |1|\, dt = \infty \quad \therefore \text{ Not BIBO Stable}$$



④ Practical Example - Integrator

$$\int_{-\infty}^{t} x_{(\tau)}\, d\tau \quad - ②$$



$$y_{(t)} = \int_{-\infty}^{t} x_{(\tau)}\, d\tau$$

- Eqn ② $\Rightarrow$ impulse respone is $h_{(t)} = u_{(t)}$

$$\text{as } \underset{\substack{\uparrow \\ \text{impulse response}}}{h_{(t)}} = \int_{-\infty}^{t} \underset{\substack{\uparrow \\ \text{input impulse}}}{\delta_{(\tau)}}\, d\tau = u_{(t)}$$

- As seen in example ③ when $h_{(t)} = u_{(t)}$, the LTI is Not BIBO stable!

⑧ Unit Step response of an LTI System

- Along with the unit impulse response $h_{(t)}$, the unit step response is also used quite often in describing the behavior of LTI systems.

- Denoted by $s(t)$ $\qquad x(t)=u(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)=s(t)$
- For LTI

$$s(t) = u(t) * h(t) = h(t) * u(t)$$

conv. comm. property.

$$= \int_{-\infty}^{t} h(\tau)\, d\tau$$

- Differentiate.

$$\boxed{h(t) = \frac{d}{dt}\, s(t)}$$

Impulse response is differential of unit step response.

- Similarly for DT

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

$$\boxed{h[n] = s[n] - s[n-1]}$$

NOTE

If $\quad x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

(I) $\quad \boxed{x'(t) * h(t) = y'(t)}$ $\quad$ where $\quad x'(t) = \frac{dx(t)}{dt}$

$$\text{LHS} = \int_{-\infty}^{\infty} h(\tau)\, \frac{d}{dt}\, x(t-\tau)\, d\tau$$

Conv. Commutative property.

$$= \frac{d}{dt} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau}_{}$$

$$= \frac{d}{dt}\, y(t)$$

(II) $\quad \boxed{u(t) * h'(t) = h(t)}$

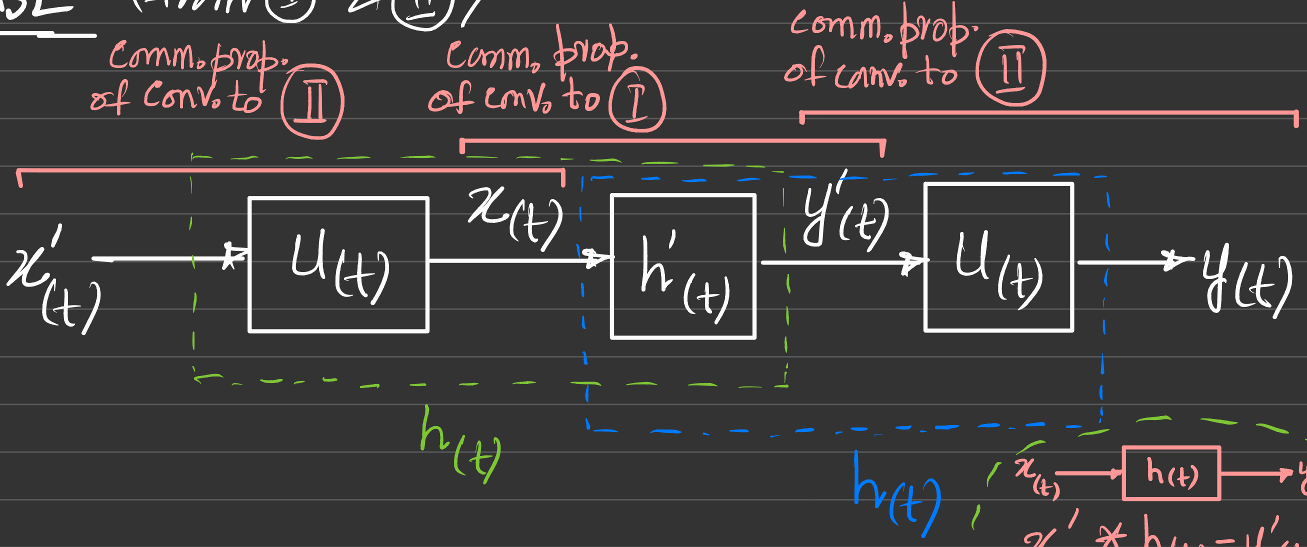
$$\text{LHS} = \int_{-\infty}^{\infty} h'(\tau)\, \underbrace{u(t-\tau)}_{=1 \text{ for } \tau<t}\, d\tau$$

Conv. commutative property.

$$= \int_{-\infty}^{t} \frac{d}{d\tau} h(\tau)\, d\tau$$

$$= h(t)$$

CASE (from (I) & (II))



$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$x'(t) * h(t) = y'(t)$$

$$h(t) \longrightarrow \boxed{x(t)} \longrightarrow y(t)$$

$$h'(t) * x(t) = y'(t)$$

CONCLUSIONS

(A) $x'(t) \longrightarrow \boxed{h(t)} \longrightarrow y'(t)$

(B) $x(t) \longrightarrow \boxed{h'(t)} \longrightarrow y'(t)$

follows from (A) & comm. property of conv.

(C) $\int_{-\infty}^{t} x(\tau)\, d\tau \longrightarrow \boxed{h'(t)} \longrightarrow y(t)$

follows from (B)

(D) $\int_{-\infty}^{t} x(\tau)\, d\tau \longrightarrow \boxed{h(t)} \longrightarrow \int_{-\infty}^{t} y(\tau)\, d\tau$

follows from (A) & (C)

(E) $x(t) \longrightarrow \boxed{\int_{-\infty}^{t} h(\tau)\, d\tau} \longrightarrow \int_{-\infty}^{t} y(\tau)\, d\tau$

follows from (D) + comm. property of conv.

Example Given $x(t) = 2e^{-3t}u(t-1) \xrightarrow{h(t)} y(t)$

$$y'_{(t)} = -3y_{(t)} + e^{-2t}u_{(t)}$$ — ②

$$h_{(t)} = ?$$

Given this!

- We know from Ⓐ $x'_{(t)} * h_{(t)} = y'_{(t)}$ — ③
- let's first find $x'_{(t)}$

$$x'_{(t)} = -6e^{-3t}u_{(t-1)} + 2e^{-3t}\delta_{(t-1)}$$

$$= -3\left[2e^{-3t}u_{(t-1)}\right] + 2e^{-3t}\delta_{(t-1)}$$

$$= -3x_{(t)} + 2e^{-3t}\delta_{(t-1)}$$ — ④

- Now

eqn. ③ & ④ ⇒ $y'_{(t)} = \left[-3x_{(t)} + 2e^{-3t}\delta_{(t-1)}\right] * h_{(t)}$

$$= -3y_{(t)} + \int_{-\infty}^{\infty} 2e^{-3\tau}\underbrace{\delta_{(\tau-1)}}_{\text{exists at } \tau=1} h_{(t-\tau)}\, d\tau$$

$$= -3y_{(t)} + 2e^{-3}h_{(t-1)} \overbrace{\int_{-\infty}^{\infty} \delta_{(\tau-1)}\, d\tau}^{=1}$$

$$= -3y_{(t)} + 2e^{-3}h_{(t-1)}$$ — ⑤

- Comparing eqn. ② & eqn. ⑤

$$2e^{-3}h_{(t-1)} = e^{-2t}u_{(t)}$$

$$h_{(t-1)} = \frac{1}{2}e^{-2t+3}u_{(t)}$$

$$\Rightarrow h_{(t)} = \frac{1}{2}e^{-2(t+1)+3}u_{(t+1)}$$

$$h_{(t)} = \frac{1}{2} e^{-2t+1} u_{(t+1)}$$